SPHERICAL FRACTIONAL INTEGRALS

BY

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1. **Introduction.** In this paper I define a fractional integral for functions on a unit m-sphere and show that it behaves in much the same way as the Riesz fractional integral in respect to theorems proved in [1].

The theorems proved in this paper are

THEOREM 1. *If* $0<\alpha<\alpha+\beta<1$ *and* $f\in\text{Lip}\ \alpha$ *then* $f_\beta\in\text{lip}\ (\alpha+\beta)$. Lip *may be replaced by* lip.

THEOREM 2. *If* $q>1$, $1+m/q>\alpha>m/q$ *then* $f_\alpha\in\text{lip}\ (\alpha-m/q)$.

THEOREM 3. *If* $0<\alpha<m/q$ *and* $f\in L^q$ *then* $f_\alpha\in L^r$ *where* $\alpha=m(1/q-1/r)$.

THEOREM 4. *If* $f\in L^q$ *then*

(a) *for* $0<\alpha<m$, $2<q<\infty$, $f_{\alpha/q}$ *is finite everywhere except possibly in a set which is of zero* β*-capacity for all* $\beta>m-\alpha$.

(b) *for* $0<\alpha<m$, $1\leqq q\leqq 2$, $f_{\alpha/q}$ *is finite everywhere except possibly in a set of zero* $(m-\alpha)$*-capacity. Both* (a) *and* (b) *are best possible.*

There is an auxiliary result in §3, Theorem A, connecting radial and surface Lipschitz behavior which is perhaps of interest in its own right. The inversion process described in §4 certainly has other possibilities.

2. **The fractional integral.** Let S be the unit m-sphere consisting of all the points $P(\xi_1, \cdots, \xi_m, \zeta)$ given by

$$\xi_1^2 + \cdots + \xi_m^2 + \zeta^2 = 1.$$

Let Q be any point and suppose that new coordinates are taken so that a point previously named P becomes the pole, i.e. the point $O(0, \cdots, 0, 1)$, then the symbol $Q-P$ will mean the point which Q becomes in the new coordinates. Further, $Q'=Q-P$ and $Q=Q'+P$ will be taken to be synonymous.

We introduce spherical polar coordinates $(\rho, \theta_1, \cdots, \theta_m)$ of the point $(\xi_1, \cdots, \xi_m, \zeta)$ by setting

$$\begin{aligned}
\xi_1 &= \rho \sin\theta_1 \cdots \qquad\qquad \sin\theta_m,\\
\xi_2 &= \rho \sin\theta_1 \cdots \sin\theta_{m-1}\cos\theta_m,\\
&\vdots\\
\xi_m &= \rho\sin\theta_1\cos\theta_2,\\
\zeta &= \rho\cos\theta_1,
\end{aligned}$$

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where $\rho>0$, $0\leqq\theta_r\leqq\pi$ for $r=1,\cdots,m-1$ and $0\leqq\theta_m\leqq 2\pi$. Points on S are characterized by $\rho=1$. In what follows points denoted by P, Q will be on S and the point $(\rho,\theta_1,\cdots,\theta_m)$ will be denoted by (ρ, P).

The distance (or, more precisely, the spherical distance) between the points $P(\xi_1,\cdots,\xi_m,\zeta)$ and $Q(\eta_1,\cdots,\eta_m,\tau)$ is given by

$$\cos|P-Q| = \xi_1\eta_1+\cdots+\xi_m\eta_m+\zeta\tau, \qquad 0\leqq|P-Q|\leqq\pi.$$

Clearly $|P-Q|=|Q-P|$ and $|P-0|=|P|=\theta$.

The surface element dP in polar coordinates is given by

$$dP = \sin^{m-1}\theta_1\cdots\sin\theta_{m-1}d\theta_m\cdots d\theta_1.$$

If $M_q^q(f)=\int_s|f(P)|^q dP<\infty$ we shall say that $f\in L^q$.

It is well known that if $f\in L$ then the Poisson integral $f(\rho, P)$ of f given by

$$f(\rho, P) = \int_S f(Q)K(\rho, Q-P)dQ$$

where

$$K(\rho, Q-P) = 2^{-1}\pi^{-(m+1)/2}\Gamma(2^{-1}(m+1))(1-\rho^2)(1-2\rho\cos|P-Q|+\rho^2)^{-(m+1)/2}$$

is harmonic for all $\rho<1$, that is, within S.

Let $f\in L$, let $f(\rho, P)$ be its Poisson integral and consider for $\alpha>0$, $0\leqq\rho<1$,

$$f_\alpha(\rho, P) = \frac{1}{\Gamma(\alpha)}\int_0^\rho(\rho-r)^{\alpha-1}f(r, P)dr$$

and $f_\alpha(P)=\lim_{\rho\to 1}f_\alpha(\rho, P)$ should this limit exist. $f_\alpha(P)$ is then said to be the αth integral of $f(P)$.

We have

$$\Gamma(\alpha)f_\alpha(\rho, P) = \int_0^\rho(\rho-r)^{\alpha-1}\int_S f(Q)K(\rho, Q-P)dQdr.$$

Now

$$K(\rho,\gamma) = \sum_{s=0}^\infty\frac{2s+m-1}{m-1}C_s^{(m-1)/2}(\cos\gamma)\rho^s$$

where the (somewhat modified) Gegenbauer polynomials $C_s^\nu(x)$ are given, for $\nu>0$, by

$$\frac{1}{2}\pi^{-\nu-1}\Gamma(\nu+1)(1-2\rho x+\rho^2)^{-\nu} = \sum_{s=0}^\infty C_s^\nu(x)\rho^s.$$

Writing $Q-P=\gamma$ we then have

$$f_\alpha(\rho, P) = \rho^\alpha \sum_{s=0}^{\infty} \frac{\Gamma(s+1)}{\Gamma(s+1+\alpha)} \frac{2s+m-1}{m-1} \rho^s \int_S f(Q) C_s^{(m-1)/2}(\cos\gamma) dQ.$$

It may be verified that if (ρ, P), Q are the points $(\xi_1, \cdots, \xi_m, \zeta)$, $(\eta_1, \cdots, \eta_m, \tau)$ then

$$\rho^2 = \xi_1^2 + \cdots + \xi_m^2 + \zeta^2, \qquad \rho\cos\gamma = \xi_1\eta_1 + \cdots + \xi_m\eta_m + \zeta\tau$$

and that, if Δ is the operator $\sum_{r=1}^m \partial^2/\partial\xi_r^2 + \partial^2/\partial\zeta^2$ then $\Delta(\rho^s C_s^{(m-1)/2}(\cos\gamma)) = 0$. It follows that $f_\alpha(\rho, P) = \rho^\alpha u_\alpha(\rho, P)$ where u_α is harmonic within S and that $f_\alpha(P) = \lim_{\rho\to 1} u_\alpha(\rho, P)$ whenever this limit exists.

We have, in fact,

For $\alpha > 0$, $f_\alpha(P)$ exists and is finite p.p. on S.

Privalov in §4 of [2] has shown that if $g(\rho, P)$ is harmonic within S and $\int_S |g(\rho, P)| dP = O(1)$ as ρ tends to 1 then $g(\rho, P)$ tends p.p. to a finite limit. Now

$$\Gamma(\alpha) \int_S |u_\alpha(\rho, P)| dP \leqq \rho^{-\alpha} \int_0^\rho (\rho - r)^{\alpha-1} \int_S |f(r, P)| dP dr.$$

Since $f \in L$ the inner integral is uniformly bounded and the result follows. This result is not best possible as Theorem 4 shows.

It is unfortunately not the case that $f_{\alpha+\beta}(P) = (f_\alpha(P))_\beta$. (A trivial calculation with $f(P) = 1$ on S shows this.) What is true is that $f_{\alpha+\beta}(\rho, P) = (f_\alpha(\rho, P))_\beta$; this is easily shown by the standard proof for the Riemann-Liouville fractional integral.

3. **Lipschitz behavior.** If f is such that

$$f(P) - f(Q) = O(|P - Q|^\alpha), \qquad 0 \leqq \alpha < 1 \quad \text{uniformly on } S$$

then we say that $f \in \text{Lip } \alpha$. If, in the above, O is replaced by o we say that $f \in \text{lip } \alpha$.

THEOREM A. *A necessary and sufficient condition that a function $f(\rho, P)$ harmonic within S is the Poisson integral of a function in* Lip α *is that*

$$f(\rho', P) - f(\rho, P) = O((\rho' - \rho)^\alpha \textit{ uniformly in}) \ P.$$

If Lip *is replaced by* lip *then O must be replaced by o.*

LEMMA 1. *For $0 \leqq \alpha < 1$, $0 < \rho < 1$,*

(a) $$\int_S |P|^\alpha \left| \frac{\partial}{\partial\rho} K(\rho, P) \right| dP \leqq B(1 - \rho)^{\alpha-1},$$

(b) $$\int_S |P|^\alpha |K(\rho, P)| dP \leqq B(1 - \rho)^\alpha$$

where B is a positive constant.

First, a straightforward calculation shows that $|\partial K/\partial\rho|$ is dominated by a positive multiple of $[(1-\rho)^2+|P|^2]^{-(m+1)/2}$. The integral in (a) is thus less than a multiple of I where

$$I = \left(\int_0^{1-\rho} + \int_{1-\rho}^{\pi}\right)\theta^\alpha[(1-\rho)^2+\theta^2]^{-(m+1)/2}\sin^{m-1}\theta d\theta = I_1 + I_2, \text{ say.}$$

Further $I_1 \leqq J_1$, $I_2 \leqq J_2$ where

$$J_1 = (1-\rho)^{-m-1}\int_0^{1-\rho}\theta^{\alpha+m-1}d\theta \quad \text{and} \quad J_2 = \int_{1-\rho}^{\pi}\theta^{\alpha-2}d\theta$$

from which (a) follows. (b) follows by integrating (a) over the range ρ to 1.

LEMMA 2. *For* $0<\lambda<1$

$$\sup_P\left|\frac{\partial}{\partial\theta_1}[(f(\rho, P) - f(\lambda\rho, P)]\right| \leqq A(1-\rho)^{-1}\sup_P|f(P) - f(\lambda, P)|.$$

We have

$$f(\rho, P) - f(\lambda\rho, P) = \int_S (f(Q) - f(\lambda, Q))K(\rho, Q-P)dQ$$

and so

$$\frac{\partial}{\partial\theta_1}(f(\rho, P) - f(\lambda\rho, P)) = \int_S [f(Q'+P) - f(\lambda, Q'+P)]\frac{\partial}{\partial\theta_1}K(\rho, \theta_1)dQ'$$

which last, in modulus, does not exceed

$$\sup_P|f(P) - f(\lambda, P)|\int_S\left|\frac{\partial}{\partial\theta_1}K(\rho, \theta_1)\right|dQ'.$$

By (a) of Lemma 1 with $\alpha=0$ the result follows.

We turn now to Theorem A.

Necessity. We may assume $\rho'>\rho$. We have

$$f(\rho', P) - f(\rho, P) = \int_S (f(Q) - f(P))(K(\rho', Q-P) - K(\rho, Q-P))dQ$$

and it follows that

$$(1) \qquad |f(\rho', P) - f(\rho, P)| \leqq A\int_S |Q|^\alpha|K(\rho', Q) - K(\rho, Q)|dQ$$

where A is a positive constant.

Either $\rho'-\rho\leqq 1-\rho'$ or $\rho'-\rho>1-\rho'$ and in the latter case $\rho'-\rho<1-\rho<2(\rho'-\rho)$.

In the first case we express the right hand side of (1) as

$$\int_S |Q|^\alpha \left| \int_\rho^{\rho'} \frac{\partial K}{\partial r} dr \right| dQ \leqq \int_\rho^{\rho'} \int_S |Q|^\alpha \left| \frac{\partial K}{\partial r} \right| dQdr = O\left\{ \int_\rho^{\rho'} (1 - r)^{\alpha-1} dr \right\}.$$

This last integral is equal to $(\rho'-\rho)(1-\bar{\rho})^{\alpha-1}$ where $\rho<\bar{\rho}<\rho'$. Since $(1-\bar{\rho})^{\alpha-1} <(1-\rho')^{\alpha-1}<(\rho'-\rho)^{\alpha-1}$ the result follows.

In the second case the righthand side of (1) is less than a multiple of

$$\int_S |Q|^\alpha K(\rho, Q)dQ + \int_S |Q|^\alpha K(\rho', Q)dQ = O\{(1 - \rho)^\alpha + (1 - \rho')^\alpha\}$$

by Lemma 1(b). This last is $O[(\rho'-\rho)^\alpha]$. This proves the necessity.

Sufficiency. Since $f(\rho', P)-f(\rho, P)=O[(\rho'-\rho)^\alpha]\to 0$ as ρ, ρ' tend to 1 uniformly in P we have, by Cauchy's convergence criterion, the existence of a bounded $f(P)$ for which $f(\rho, P)$ tends to $f(P)$ uniformly in P as ρ tends to 1.

Given ρ, choose $\rho'>\rho$ such that, for all P,

$$|f(P) - f(\rho', P)| < (1 - \rho)^\alpha.$$

Then

$$|f(P) - f(\rho, P)| \leqq |f(P) - f(\rho', P)| + |f(\rho', P) - f(\rho, P)|$$

which does not exceed $(1-\rho)^\alpha+O[(\rho'-\rho)^\alpha]=O[(1-\rho)^\alpha]$.

Let $\rho_n=1-2^{-n}$, write $\phi_n(P)=f(\rho_{n+1}, P)-f(\rho_n, P)$ and choose N so that $2^{-N-1}\leqq|P|\leqq 2^{-N}$. We then have $|f(Q+P)-f(Q)|$ dominated by

$$|f(Q + P) - f(\rho_N, Q + P)| + |f(Q) - f(\rho_N, Q)| + |f(\rho_N, Q + P) - f(\rho_N, Q)|$$

and this last does not exceed

$$O(2^{-N\alpha}) + \sum_{n=0}^{N-1} |\phi_n(Q + P) - \phi_n(Q)|. \tag{2}$$

Choose the pole on the sphere at Q and coordinates so that P is $(t, 0, \cdots, 0)$. Then

$$|\phi_n(Q + P) - \phi_n(Q)| = |\phi_n(P) - \phi_n(O)| \leqq \int_0^t \left| \frac{\partial \phi_n}{\partial \theta_1} \right| d\theta_1.$$

This last, by Lemma 2, is dominated by

$$At(1 - \rho_{n+1})^{-1} \sup_P |f(P) - f(\lambda_n, P)|$$

where $\lambda_n=\rho_n/\rho_{n+1}$. It follows that

$$|\phi_n(Q + P) - \phi_n(Q)| \leqq A2^{-N}2^{n+1}(1 - \lambda_n)^\alpha \leqq A2^{1-N}2^{(1-\alpha)n}.$$

Thus, finally,

$$\sum_{n=0}^{\infty} \left| \phi_n(Q+P) - \phi_n(Q) \right| \leqq A2^{1-N} \sum_{n=0}^{N} 2^{(1-\alpha)n} = O(2^{-\alpha N}).$$

This, in conjunction with (2), gives the O-part of Theorem A. The proof of the o-part is an obvious adaptation of the above.

The proof of Theorem 1 now follows readily. If $f(P) \in \mathrm{Lip}\ \alpha$, say, $f(\rho, P)$ is, by Theorem A, a function in Lip α with respect to ρ. The standard proof for Riemann-Liouville integrals [3, §9.81] shows that $f_\beta(\rho, P)$ is, qua ρ, in Lip $(\alpha+\beta)$. It follows that the same holds for $u_\beta(\rho, P)$. Since u_β is harmonic we may apply Theorem A to show $f_\beta(P)$ in Lip $(\alpha+\beta)$. A similar argument shows the lip part of the theorem.

We turn now to Theorem 2. Let $S(\delta)$ be the spherical cap $\theta_1 \leqq \delta$. Since $f \in L^q$ we may, given $\epsilon > 0$, choose δ so that $\int_{S(\delta)} |f(Q)|^q dQ < \epsilon^q$. Now

$$\frac{\partial f}{\partial \rho} = \left(\int_{S(\delta)} + \int_{cS(\delta)} \right) f(Q+P) \frac{\partial K}{\partial \rho} dQ = I_1 + I_2.$$

and $I_1 < \epsilon M_{q'}(\partial K/\partial \rho)$. Further, $M_{q'}^{q'}$ is dominated by a constant multiple of

$$\int_0^\pi \theta^{m-1}[(1-\rho)^2 + \theta^2]^{-(m+1)q'/2} d\theta$$

$$\leqq (1-\rho)^{m-(m+1)q'} \int_0^\infty t^{m-1}(1+t^2)^{-(m+1)q'/2} dt$$

so that

$$I_1 = o[(1-\rho)^{-m/q-1}].$$

Next

$$I_2 \leqq M_q(f) \left\{ \int_{cS(\delta)} \left| \frac{\partial K}{\partial \rho} \right|^{q'} dQ \right\}^{1/q'}$$

and the second integral does not exceed a multiple of

$$\int_\delta^\pi \theta^{m-1}[(1-\rho)^2 + \theta^2]^{-(m+1)q'/2} d\theta$$

$$= (1-\rho)^{m-(m+1)q'} \int_{\delta/(1-\rho)}^\infty t^{m-1}(1+t^2)^{-(m+1)q'/2} dt$$

so that $I_2 = o[(1-\rho)^{-m/q-1}]$. Consequently $\partial f/\partial \rho = o[(1-\rho)^{-m/q-1}]$.

We may, without loss of generality, assume $f(0, P) = 0$ and then integrating the integral expression for $f_\alpha(\rho, P)$ by parts and then differentiating under the integral sign, we have

$$\Gamma(\alpha) \frac{\partial}{\partial \rho} (\bar{f}_\alpha) = \int_0^\rho (\rho - r)^{\alpha-1} \frac{\partial f}{\partial r} dr = o\left\{ \int_0^\rho (\rho - r)^{\alpha-1}(1-r)^{-m/q-1} dr \right\}$$

so that $\partial/\partial\rho(f_\alpha)=o[(1-\rho)^{\alpha-m/q-1}]$. Consequently

$$f_\alpha(\rho', P) - f_\alpha(\rho, P) = \int_\rho^{\rho'} \frac{\partial}{\partial r}(f_\alpha) = o[(\rho'-\rho)^{\alpha-m/q}]$$

so, by Theorem A, since now $u_\alpha(\rho, P)\in\text{Lip}\,(\alpha-m/q)$, qua ρ, we have $f_\alpha(P)\in\text{Lip}\,(\alpha-m/q)$ thus obtaining Theorem 2.

4. **The inversion process.** Invert the $(m+1)$-space in which S is embedded taking $(0, \cdots, 0, -1)$ as centre of inversion and a radius of inversion 2. Let $X(x_1, \cdots, x_m, z)$ denote the transform of $P(\xi_1, \cdots, \xi_m, \zeta)$ referred to axes parallel to those to which P is referred but with origin at $\xi_r=0$, $r=1, \cdots, m$, $\zeta=1$.

The equations of transformation are then

$$\xi_r = 4R^{-2}x_r, \qquad r = 1, \cdots, m; \qquad 1+\zeta = 4R^{-2}(z+2)$$

where $R^2=x_1^2+\cdots+x_m^2+(z+2)^2$.

Under the inversion S transforms into the plane $z=0$ which we shall hereafter denote by S' and the region $\theta_1<\alpha$ of S transforms into the interior of the "circle"

$$x_1^2+\cdots+x_m^2 = (2\tan\frac{1}{2}\alpha)^2.$$

When $(\xi_1, \cdots, \xi_m, \zeta)$ is on S then $R^2=x_1^2+\cdots+x_m^2+4$ and $\zeta=(8-R^2)R^{-2}$. It follows that if $P(\xi_1, \cdots, \xi_m, \zeta)$ and $Q(\eta_1, \cdots, \eta_m, \tau)$ transform into $X(x_1, \cdots, x_m)$ and $Y(y_1, \cdots, y_m)$ then

$$\cos|Q-P| = [16(x_1y_1+\cdots+x_my_m)+(8-R_X^2)(8-R_Y^2)]R_X^{-2}R_Y^{-2}$$

where $R_X^2=x_1^2+\cdots+x_m^2+4$ and $R_Y^2=y_1^2+\cdots+y_m^2+4$. So

$$\sin 2^{-1}|Q-P| = 4|X-Y|^2R_X^{-2}R_Y^{-2} \text{ where } |X-Y|^2 = \sum_{r=1}^m (x_r-y_r)^2.$$

Consequently

$$|Q-P| \text{ lies between constant multiples of } (R_XR_Y)^{-1}|X-Y|.$$

It is a straightforward matter to verify that the angle between the normal to S at P and the ray OP from the center of inversion is equal to the angle between the normal to S' at X and the ray OX (which is identical with OP). It follows that

$$(OP)^{-m}dP = (OX)^{-m}dX \text{ so that } dP = 4^mR^{-2m}dX.$$

5. **Proof of Theorem 3.**

LEMMA 3. *For $0<\alpha<m$ and positive $f(P)$*

$$A\int_S|Q-P|^{\alpha-m}f(Q)dQ \leqq f_\alpha(P) \leqq B\int_S|Q-P|^{\alpha-m}f(Q)dQ$$

where A and B are positive constants.

I show first that if $\Delta = \Delta(\rho, \gamma) = 1 - 2\rho \cos \gamma + \rho^2$ then

$$A\Delta^{(\alpha-m)/2} \leqq \int_0^\rho (\rho - r)^{\alpha-1} K(r, \gamma) dr \leqq B\Delta^{(\alpha-m)/2} \text{ for } 1 > \rho > 1/2. \tag{3}$$

The integral in the center lies between positive multiples of

$$\int_0^\rho (\rho - r)^{\alpha-1}(1 - r)[(1 - r)^2 + \gamma^2]^{-(m+1)/2} dr. \tag{4}$$

If $1 - \rho \geqq \gamma$ (4) lies between positive multiples of

$$\int_0^\rho (\rho - r)^{\alpha-1}(1 - r)^{-m} dr = (1 - \rho)^{\alpha-m} \int_1^{1/(1-\rho)} (t - 1)^{\alpha-1} t^{-m} dt$$

so that in this case the center integral lies between multiples of $(1 - \rho)^{\alpha-m}$.

If $1 - \rho < \gamma \leqq 1$ (4) lies between positive multiples of

$$\int_0^{1-\gamma} (\rho - r)^{\alpha-1}(1 - r)^{-m} dr + \gamma^{-m-1} \int_{1-\gamma}^\rho (\rho - r)^{\alpha-1}(1 - r) dr = T_1 + T_2, \text{ say.}$$

Now

$$T_1 = (1 - \rho)^{\alpha-m} \int_{\gamma/(1-\rho)}^{1/(1-\rho)} (t - 1)^{\alpha-1} t^{-m} dt$$

and

$$T_2 = \gamma^{-m-1}(1 - \rho)^{\alpha+1} \int_1^{\gamma/(1-\rho)} (t - 1)^{\alpha-1} t\, dt.$$

Both these are $O(\gamma^{\alpha-m})$ but both cannot simultaneously be $o(\gamma^{\alpha-m})$.

If $\gamma > 1$ (4) is clearly $O(\gamma^{\alpha-m})$ and not $o(\gamma^{\alpha-m})$.

Thus the center integral lies between positive multiples of min $[(1 - \rho)^{\alpha-m}, \gamma^{\alpha-m}]$. Since $\Delta^{(\alpha-m)/2}$ also lies between positive multiples of this quantity (3) follows.

Now

$$\Gamma(\alpha) f_\alpha(\rho, P) = \int_S f(Q) \int_0^\rho (\rho - r)^{\alpha-1} K(r, Q - P) dr dQ$$

so, by (3), it lies between positive multiples of $\int_S f(Q)\Delta^{(\alpha-m)/2}(\rho, Q - P) dQ$. Using Fatou's Lemma on the one hand and trivially on the other we have Lemma 3.

Denote the spherical cap of radius $\pi/4$ and center P by K_P and the cap of radius $\pi/2$ and with the pole 0 as center, i.e. the upper hemisphere, by

H. Now the union of all K for which $P \in H$ is itself contained in the region K given by $\theta_1 \leqq 3\pi/4$.

In what follows we shall denote the transform of any set M in S into S' by M'. We also write

$$\mathfrak{M}_q^q(F) = \int_{S'} |F(X)|^q dX.$$

In proving Theorem 3 we may assume $f(P)$ positive. It then follows that

$$\begin{aligned}\int_S |Q - P|^{\alpha-m} f(P)dP &= \left(\int_{K_P} + \int_{cK_P}\right) |Q - P|^{\alpha-m} f(P)dP \\ &\leqq \int_{K_P} |Q - P|^{\alpha-m} f(P)dP + BM_q(f).\end{aligned}$$

Let P invert into X and let $f(P)$ become $F(X)$. Then

$$\int_{K_P} |Q - P|^{\alpha-m} f(Q)dQ \leqq A \int_{K'_P} |X - Y|^{\alpha-m} (R_X R_Y)^{m-\alpha} F(Y) R_Y^{-2m}\, dY.$$

Since $K_P \subset K$ we have $K'_P \subset K'$ and in K' R_X and R_Y lie between 2 and $2(1+\tan^2 3\pi/8)^{1/2}$ so that the last integral does not exceed a constant multiple of $\int_{K'_P} |X - Y|^{\alpha-m} F(X)dX$.

Thus, to sum up

$$f_\alpha(P) \leqq B \int_{K'} |X - Y|^{\alpha-m} F(X)dX + cM_q(f). \tag{5}$$

Next,

$$\int_K |f(P)|^q dP = 4^m \int_{K'} |F(X)|^q R_X^{-2m}\, dX < \infty$$

so that the function $G(X)$ which equals $F(X)$ in K' and vanishes elsewhere is in L^q over S'. Theorem 3 of [1] states that if

$$G_\alpha(X) = K_m^{-1} \int_{S'} |X - Y|^{\alpha-m} G(X)dX, \qquad K_m \text{ a constant,}$$

then $\mathfrak{M}_r(G_\alpha) \leqq A\mathfrak{M}_q(G)$ where $\alpha = m(1/q - 1/r)$. Since, by (5),

$$f_\alpha(P) \leqq B' G_\alpha(X) + CM_q(f)$$

and $M_q^q(G)$ does not exceed a multiple of $\int_K |f(P)|^q dP \leqq M_q^q(f)$ we have $\{\int_H |f_\alpha(P)|^r dP\}^{1/r}$ dominated by a multiple of $M_q(f)$. A similar argument for cH then gives Theorem 3.

6. Some preliminaries about capacity. Let $\mu(M)$ be a non-negative additive set function defined for all Borel sets in S. If $\mu(S) = 1$ we say that μ is a distribution. If, further, $\mu(M)=1$ we say that the distribution is concentrated on M.

Suppose M is a set on which a distribution μ is concentrated and consider, for $\beta>0$,

$$V_\beta = \sup_{P\in S} \int_S |Q-P|^{-\beta} d\mu(Q). \tag{6}$$

If V_β is finite we say M is of positive β-capacity. Otherwise M is said to be of zero β-capacity. Clearly, if M is of positive β-capacity it is of positive γ-capacity for all $\gamma<\beta$. Also, clearly, if M is of zero β-capacity it is of zero γ-capacity for all $\gamma>\beta$.

Similar considerations apply to S'. Given any set M' and a distribution ν concentrated on it we consider

$$V'_\beta = \sup_{X\in S'} \int_{S'} |X-Y|^{-\beta} d\nu(Y). \tag{7}$$

Definitions are framed and consequences drawn as in the last paragraph.

A distribution μ on S induces a distribution ν on S' as follows. Let $\mu(M)=\nu(M')$. If μ is concentrated on M ν is concentrated on M' and vice versa.

Lemma 4. *M is of zero β-capacity if and only if M' is so. Also M is of positive β-capacity if and only if M' is so.*

If. Let μ be concentrated on M and so ν concentrated on M'. Now

$$\int_S |P-Q|^{-\beta} d\mu(Q) > \int_{S'} |X-Y|^{-\beta} R_X^\beta R_Y^\beta d\overset{\beta}{\nu}(Y) > \int_{S'} |X-Y|^{-\beta} d\nu(Y)$$

so that $V_\beta > V'_\beta$. If M' is of zero β-capacity then clearly so is M.

Only if. Let $M'_n = M' \cap \text{set } [X\in S';\ R_X \leqq n]$. Then $M'_n \subset M'$ so $M_n \subset M$. If M is of zero β-capacity so is M_n. Further

$$V'_\beta(M'_n) \geqq n^{-2\beta} V_\beta(M_n) = +\infty$$

so M'_n is of zero β-capacity. Finally $M' = \bigcup_{n=1}^\infty M'_n$ and so [4, p. 50] is of zero β-capacity. The second part of the lemma is now trivial.

7. **Proof of Theorem 4.** Let $f\in L^q$ and let M be the set in which $f_{\alpha/q}$ is infinite. We may suppose that f is positive so that, by (5),

$$\int_{K'} |X-Y|^{\alpha/q-m} F(X)dX$$

i.e., $G_{\alpha/q}(x)$ (defined in §5) is infinite in M'. But $G(x)\in L^q$ so that, by Theorem 4 of [1], M' is of zero β-capacity for all $\beta>m-\alpha$ for $q>2$, and is of zero $(m-\alpha)$-capacity for $1\leqq q\leqq 2$. Since, by Lemma 4, M has the same capacity behavior as M' Theorem 4 is proved.

To show Theorem 4 best possible we note that in §6 of [1] a positive function $F(X) \in L^q$ and a set M' is constructed such that $F_{\alpha/q}(X)$ is infinite in M' and M' is of positive β-capacity (where $\beta = m - \alpha$ when $1 \leqq q \leqq 2$ and β is any number greater than $m - \alpha$ when $q > 2$).

Let $F(X)$ transform to $f(P)$ on S. Since $F > 0, f > 0$ and

$$F_{\alpha/q}(X) < \int_{S'} | X - Y |^{\alpha/q - m} R_x^{m-\alpha/q} R_y^{m-\alpha/q} F(Y) dY < \int_S | P - Q |^{\alpha/q - m} f(Q) dQ.$$

Lemma 3 shows that $f_{\alpha/q}(P)$ is infinite in M and this has the same capacity behavior as M'. Theorem 4 is thus shown best possible.

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